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DYNAMICAL CONSEQUENCES OF A FREE INTERVAL: MINIMALITY, TRANSITIVITY, MIXING AND TOPOLOGICAL ENTROPY

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Dedicated to our teacher, friend and colleague Alfonz Haviar on the occasion of his retirement.

Abstract. We study dynamics of continuous maps on compact metrizable spaces containing a free interval (i.e., an open subset homeomorphic to an open interval). A special attention is paid to relationships between topological transitivity, weak and strong topological mixing, dense periodicity and topological entropy as well as to the topological structure of minimal sets. In particular, a trichotomy for minimal sets and a dichotomy for transitive maps are proved.

1. Introduction

One-dimensional dynamics became an object of wide interest in the middle of 1970's, some 10 years after Sharkovsky's theorem, when chaotic phenomena were discovered by Li and Yorke in dynamics of interval maps. As general references one can recommend the monographs [CE80], [BC92], [ALM00] and [dMvS93] (several motivations for studying one-dimensional dynamical systems are discussed in the introduction of [dMvS93]). The interval and circle dynamics are well understood. To extend/generalize the results to graph maps is sometimes quite easy, sometimes extremely difficult (for instance the characterization of the set of periods for graph maps is known only in very special cases). A good, more than 50 pages long survey of some topics in the dynamics of graph maps can be found in an appendix in [ALM00] (the second edition). Also the paper [Bl84] and the paper [Bl86] with its two continuations under the same title are a must for everybody who wishes to study the dynamics of graph maps.

When working in one-dimensional topological dynamics it is natural to try to extend results from the interval to more general spaces. One usually extends them (or slight modifications of them) first to the circle/trees and then to general graphs (then perhaps also to special kinds of dendrites or to other one-dimensional continua; in the case of general dendrites often a counterexample can be found). The present paper suggests that sometimes another approach can be more fruitful — we show that some important facts from the topological dynamics on the interval/circle work on much more general spaces than graphs, namely on spaces containing an *open* part looking like an interval (we will call it a *free interval*). It seems that the first result indicating that the presence of a free interval might have important dynamical consequences for the whole space was obtained as early as 1988 in [Ka88], for two other results see [AKLS99] and [HKO11]. However, they were isolated in a sense and apparently have not attracted much attention; a systematic study of the influence of a free interval on dynamical properties of a space has not been done yet. Based on the main results of the present paper, we believe that namely the class of spaces with a free interval is a natural candidate for possible extension of classical results of one-dimensional topological dynamics. Of course, not all of them can be carried over from the interval/circle to spaces with a free interval (and then trees/graphs naturally enter the scene as candidates for possible extension). However,

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we hope that the elegance of the main results of the present paper (see Theorems A, B and C) justifies our belief.

Recall the terminology which is being used to describe spaces studied in this paper.

Definition 1. Let X be a topological space. We say that J is a *free interval of* X if it is an open subset of X homeomorphic to an open interval of the real line.

Definition 2. Let X be a connected topological space and J be a free interval of X. We say that J is a *disconnecting interval of* X if $X \setminus \{x\}$ has exactly two components for every point $x \in J$.

The definition of a disconnecting interval is taken from [AKLS99]. One can suggest several other "natural" definitions of a disconnecting interval. However, we warn the reader that only some of them are equivalent in the setting of general connected Hausdorff topological spaces. We postpone a discussion on this to Appendix 2.

Throughout the paper, by an *interval* in X we mean any nonempty subset of X homeomorphic to a (possibly degenerate) interval in $\mathbb{R}$. Of course, all free intervals are intervals. An open subinterval of a free/disconnecting interval is also free/disconnecting.

An *arc* A is a homeomorphic image of $[0,1]$. The closure of a free interval of X need not be an arc (say, this closure may look like the topologist's sine curve). If it *is* an arc, it is called a *free arc* of X. Notice that a space X contains a free arc if and only if it contains a free interval.

Here are all the results on the dynamics of continuous maps on general spaces with a free or a disconnecting interval, which we have found in the literature.

- No Peano continuum with a free arc admits an expansive homeomorphism ([Ka88]). For generalizations see [MS07], [SW09].
- If X is a connected space with a disconnecting interval and a continuous map $f : X \to X$ is transitive then the set of all periodic points of f is dense in X ([AKLS99]).
- Let X be a compact metrizable space with a free interval. Then every totally transitive continuous map $f : X \to X$ with dense periodic points is strongly mixing ([HKO11]).

We suggest to study the dynamics of continuous maps on spaces with a free or a disconnecting interval systematically and compare the results with those working on graphs or trees. For the present paper we have chosen those problems which seemed to us most important/interesting. The obtained results are potentially applicable in the study of dynamics on some classes of one-dimensional continua and some of them are even surprisingly nice.

Let us now summarize the main results of this paper. The first two theorems deal with minimal systems and minimal sets, the third one is a dichotomy for transitive maps. Note that all the maps in the paper are assumed to be continuous. For definitions see Sections 2 and 3 (here we recall only those which are not widely known).

It is well known that a graph admits a minimal map if and only if it is a finite union of disjoint circles (see [Bl84], cf. [BHS03]). Our first result generalizes this fact.

Theorem A (Minimal spaces with a free interval). *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a minimal map. Then X is a disjoint union of finitely many circles, $X = \bigoplus_{i=0}^{n-1} \mathbb{S}^1_i$, which are cyclically permuted by f and, on each of them, f^n is topologically conjugate to the same irrational rotation.*

Also characterization of minimal sets on graphs is well known: these are finite sets, Cantor sets and finite disjoint unions of circles, see [BHS03]. In [BDHSS09] this was generalized to local dendrites. In that connection the notion of a cantoroid was introduced. According to [BDHSS09], a *cantoroid* is a compact metrizable space without isolated points in which degenerate (connected) components are dense.

The following theorem describes the minimal sets which intersect a free interval of a space.

Theorem B (Trichotomy for minimal sets). *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a continuous map. Assume that M is a minimal set for f which intersects J. Then exactly one of the following three statements holds.*

(1) *M is finite.*
(2) *M is a nowhere dense cantoroid.*
(3) *M is a disjoint union of finitely many circles.*

Note that none of these three conditions can be removed. Already on the circle each of the three cases can occur; here in the case (2) M is a Cantor set. Moreover, if X is a (local) dendrite containing a free interval as well as a non-degenerate nowhere dense subcontinuum then there is a cantoroid different from a Cantor set which intersects that free interval and contains that subcontinuum and so by [BDHSS09] it is a minimal set for some continuous selfmap of X. Hence, in the case (2) we cannot replace "cantoroid" by "Cantor set".

Obviously, in the case (2), $M \cap J$ is a union of countably many Cantor sets (sometimes such a set is called a *Mycielski set*) and if $M \subseteq J$ then M is a Cantor set. In the case (3), M contains the whole free interval J and by Theorem A, applied to $f|_M$, we get that M is a disjoint union of finitely many circles which are cyclically permuted by f. Then, on each of these circles, the corresponding iteration of f is topologically conjugate to the same irrational rotation.

To state our next theorem we recall the following notion. If $\mathcal{D} = (D_0, \dots, D_{n-1})$ is a regular periodic decomposition for f (see Section 3) we say, according to [Ba97], that f is *strongly mixing relative to* $\mathcal{D}$ if f^n is strongly mixing on each of the sets D_i. Also, we say that f is *relatively strongly mixing* if it is strongly mixing relative to some of its regular periodic decompositions.

Theorem C (Dichotomy for transitive maps). *Let X be a compact metrizable space with a free interval and let $f : X \to X$ be a transitive map. Then exactly one of the following two statements holds.*

(1) *The map f is relatively strongly mixing, non-invertible, has positive topological entropy and dense periodic points.*
(2) *The space X is a disjoint union of finitely many circles, $X = \bigoplus_{i=0}^{n-1} \mathbb{S}^1_i$, which are cyclically permuted by f and, on each of them, f^n is topologically conjugate to the same irrational rotation.*

Three obvious remarks to Theorem C can be made. First, if X is a continuum with a *disconnecting* interval then necessarily the case (1) holds. Second, if f is *totally* transitive then X is a continuum (otherwise X has $k \geq 2$ components permuted by f, a contradiction), in the case (1) the map f is strongly mixing (if not then the RPD from the definition of relative strong mixing has $m \geq 2$ elements and f^m is not transitive) and in the case (2) the space X is a circle. Third, none of the two conditions in Theorem C is superfluous. To see it, just consider the tent map on the interval and an irrational rotation of the circle.

Kwietniak in [Kw11] independently obtained the following result which follows from our Theorem C: Any weakly mixing map on a compact metrizable space with a free interval is strongly mixing, has dense periodic points and positive entropy.

By [Bl84], on *graphs* the following dichotomy, stronger than that in our Theorem C, is true: A transitive system on a (not necessarily connected) graph either has the relative specification property (hence also all the properties from the case (1)) or the case (2) from Theorem C holds. It is a challenge to solve the problem whether this stronger dichotomy is true for every compact metrizable space with a free interval.

For some other results which are worth of mentioning and are not covered by Theorems A–C, see Theorems 19, 20 and Corollary 30.

Let us state some applications of our main results.

The Warsaw circle is one of the simplest continua which are not locally connected. The dynamics on this particular space has been studied since 1996, see [XYZH96]. The main result of the recent paper [ZPQY08] says that every transitive map on the Warsaw circle W has a horseshoe (hence positive topological entropy) and dense periodic points and is strongly mixing. Our Theorem C gives, for granted, a result which is on one hand only slightly weaker and, on the other hand, works

on a whole class of spaces including the Warsaw circle. Namely if $X \neq \mathbb{S}^1$ is a continuum with a free interval then every transitive map on X has positive topological entropy and dense periodic points and every totally transitive map on X is strongly mixing.

Baldwin in [Ba01] asked whether every transitive map on a dendrite has positive topological entropy. The problem is still open but notice that our Theorem C implies that the answer is positive for dendrites whose branch points are not dense.

The paper is organized as follows. In Section 2 we recall some definitions and known facts from topological dynamics. In Section 3 we apply the theory of regular periodic decompositions for transitive maps, developed by Banks in [Ba97] in the setting of general topological spaces, to spaces with free intervals. In Section 4 we study connections between recurrent and periodic points on spaces with free intervals. In Section 5 we prove some conditions sufficient for the density of (eventually) periodic points. Section 6 deals with minimal systems and minimal sets on spaces with free intervals; it contains proofs of Theorems A and B. Then in Sections 7, 8 and 9 we study, respectively, dense periodicity, topological entropy and strong mixing for transitive maps. The proof of Theorem C is contained in Section 9. A proof of Theorem 20, which is a generalization of a theorem from [MS09], is given in Appendix 1. Finally, in Appendix 2 we discuss relations between several "natural" definitions of a disconnecting interval, see Proposition 44. Moreover, the main results of both appendices are used in the proof of Lemma 21.

2. Preliminaries

Here we briefly recall all the notions and results which will be needed in the rest of the paper.

We write $\mathbb{N}$ for the set of positive integers $\{1, 2, 3, \dots\}$ and I for the unit interval $[0, 1]$. A *space* means a topological space. If X is a connected space and $x \in X$ is a cut point of X (i.e. $X \setminus \{x\}$ is not connected) we also say that *x cuts X* (by saying that *x cuts X into two components* we mean that $X \setminus \{x\}$ has exactly two components). A *continuum* is a connected compact metrizable space. By $\operatorname{Int} A$, $\overline{A}$, ∂A and $\operatorname{card} A$ we denote the interior, closure, boundary and cardinality of A. For definitions of a cantoroid, a free interval/arc and a disconnecting interval see Section 1. If J is a free interval/arc then we always assume that one of two natural orderings (induced by usual orderings on a real interval) is chosen and denoted by $\prec$. We will use the usual notations for subintervals of J, say we write $[a, b) = \{x \in J : \ a \preceq x \prec b\}$ for $a \prec b$ in J. Throughout the paper no distinction is made between a point x and the singleton $\{x\}$.

A *(discrete) dynamical system* is a pair (X, f) where X is a topological space and $f : X \to X$ is a (possibly non-invertible) continuous map. The iterates of f are defined by $f^0 = \mathrm{Id}_X$ (the identity map on X) and $f^n = f^{n-1} \circ f$ for $n \geq 1$. The *orbit* of x is the set $\mathrm{Orb}_f(x) = \{f^n(x) : \ n \geq 0\}$. A point $x \in X$ is a *periodic point* of f if $f^n(x) = x$ for some $n \in \mathbb{N}$. The smallest such n is called the *period* of x. If $f^m(x)$ is periodic for some $m \in \mathbb{N}$ we say that x is *eventually periodic*. A point x is *recurrent* if for every neighborhood U of x there are arbitrarily large n with $f^n(x) \in U$. By $\mathrm{Per}(f)$ or $\mathrm{Rec}(f)$ we denote the set of all periodic or recurrent points of f, respectively.

A system (X, f) is *minimal* if every orbit is dense. A set $A \subseteq X$ is *minimal* for f if it is nonempty, closed, *f-invariant* (i.e. $f(A) \subseteq A$) and $(A, f|_A)$ is a minimal system. A system (X, f) is called *totally minimal* if (X, f^n) is minimal for every $n = 1, 2, \dots$.

A dynamical system (X, f) is *(topologically) transitive* if for every non-empty open sets $U, V \subseteq X$ there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. A point whose orbit is dense is called a *transitive point*; points which are not transitive are called *intransitive*. The set of transitive or intransitive points of f is denoted by $\mathrm{Tr}(f)$ or $\mathrm{Intr}(f)$, respectively. If (X, f^n) is transitive for all $n \in \mathbb{N}$ we say that (X, f) is called *totally transitive*. If $(X \times X, f \times f)$ is transitive then (X, f) is called *(topologically) weakly mixing*. It is well known that if X is a compact metrizable space and (X, f) is weakly mixing then for every $n \geq 1$ the system $(X \times \cdots \times X, f \times \cdots \times f)$ (n-times) is topologically transitive. A system (X, f) is *(topologically) strongly mixing* provided for every non-empty open sets $U, V \subseteq X$ there is $n_0 \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for every $n \geq n_0$. Instead of saying that a system (X, f)

has some of the defined properties (minimality, transitivity, $\dots$) we also say that the map f itself has this property.

Finally, the *topological entropy* of (X, f) will be denoted by $h(f)$. We assume that the reader is familiar with Bowen's definition of topological entropy which uses the notion of (n, ε)-separated sets; see for instance [ALM00].

Lemma 3 ([Ba97])**.** *Let X be a topological space and $f : X \to X$ a continuous map. If f is totally transitive and has dense set of periodic points, then f is weakly mixing.*

Lemma 4 ([AKLS99])**.** *Let X be a connected topological space with a disconnecting interval J and let $f : X \to X$ be a continuous map. Assume that there exist $x, y \in J$ and $n, m \ge 1$ such that $f^n(x), f^m(y) \in J$, $f^n(x) \prec x$ and $f^m(y) \succ y$. Then f has a periodic point in the convex hull of $\{x, y, f^n(x), f^m(y)\}$.*

Lemma 5 ([AKLS99])**.** *Let X be a connected topological space with a disconnecting interval and let $f : X \to X$ be a transitive map. Then the set of all periodic points of f is dense in X.*

Our Theorem C is stronger than the following result from [HKO11]; however, we will use it in the course of proving Theorem C.

Lemma 6 ([HKO11])**.** *Let X be a compact metrizable space with a free interval. Then every totally transitive map $f : X \to X$ with dense periodic points is strongly mixing.*

We will use several topological properties of minimal systems which we summarize in the following lemmas.

A set $G \subseteq X$ is said to be a *redundant open set for a map* $f : X \to X$ if G is nonempty, open and $f(G) \subseteq f(X \setminus G)$ (i.e., its removal from the domain of f does not change the image of f).

Lemma 7 ([KST01])**.** *Let X be a compact Hausdorff space and $f : X \to X$ continuous. Suppose that there is a redundant open set for f. Then the system (X, f) is not minimal.*

Lemma 8 ([KST01])**.** *Let X be a compact Hausdorff space and let $f : X \to X$ be a minimal map. Then f is feebly open, i.e. f sends nonempty open sets to sets with nonempty interior.*

Lemma 9 ([KST01])**.** *Let X be a compact metrizable space and let $f : X \to X$ be a minimal map. Then the set $\{x \in X : \operatorname{card} f^{-1}(x) = 1\}$ is a G_δ-dense set in X.*

Notice that a minimal map f on a compact Hausdorff space X is necessarily surjective and so $f^{-1}(x)$ is nonempty for every $x \in X$.

Minimal maps preserve several important topological properties of sets both forward and backward. We will explicitly use the following result.

Lemma 10 ([KST01])**.** *Let X be a compact Hausdorff space and let $f : X \to X$ be a minimal map. If R is a residual subset of X then so is $f(R)$.*

If (X, f) is a dynamical system and $x \in X$ then any subset $\{x_n : n \ge 0\}$ of X satisfying $x_0 = x$ and $f(x_{n+1}) = x_n$ $(n \ge 0)$ is called a *backward orbit* of x (under f). We will use the following fact (see the proof of [KST01, Theorem 2.8] or [Ma11]; in fact also the converse is true – for continuous selfmaps of compact metrizable spaces the density of all backward orbits implies minimality).

Lemma 11 ([KST01])**.** *Let X be a compact metrizable space and let $f : X \to X$ be a continuous map. If f is minimal then all backward orbits are dense.*

The following result easily follows from [Ye92, Theorem 3.1].

Lemma 12 ([Ye92])**.** *Let X be a compact Hausdorff space and let $f : X \to X$ be a minimal map. If X is connected then f is totally minimal.*

The following classical result is an immediate consequence of [Ki58] and [Si92] (proved also in [KS97]).

Lemma 13 ([Ki58], [Si92]). *Let X be a compact metrizable space without isolated points and let $f : X \to X$ be a continuous map. Then one of the following holds:*

1. $\mathrm{Tr}(f) = \emptyset$ *and* $\mathrm{Intr}(f) = X$*;*
2. $\mathrm{Tr}(f)$ *is* G_δ*-dense and* $\mathrm{Intr}(f)$ *is either empty (i.e. the system is minimal) or dense (then the system is transitive non-minimal).*

Lemma 14 ([D32]). *If f is a transitive homeomorphism of a circle then it is conjugate to an irrational rotation.*

3. Regular periodic decompositions for transitive maps

In this section we study regular periodic decompositions for transitive maps on spaces with a free interval. We begin by reviewing some results from [Ba97].

Let X be a topological space. Recall that a set $D \subseteq X$ is *regular closed* it it is the closure of its interior or, equivalently, if it equals the closure of an open set.

Now let $f : X \to X$ be a continuous map. A *regular periodic decomposition* (briefly RPD) for f is a finite sequence $\mathcal{D} = (D_0, \dots, D_{m-1})$ of regular closed subsets of X covering X such that $f(D_i) \subseteq D_{i+1 (\mathrm{mod}\, m)}$ for $0 \le i \le m-1$ and $D_i \cap D_j$ is nowhere dense in X for $i \ne j$. The integer m is called the *length* of $\mathcal{D}$. From the latter condition in the definition and from the fact that the boundary of a regular closed set is nowhere dense, we get, respectively,

(RPD1) $\mathrm{Int}(D_i) \cap D_j = \emptyset$ for $i \ne j$;
(RPD2) the boundary of each D_i is nowhere dense.

Now let $\mathcal{D} = (D_0, \dots, D_{m-1})$ be an RPD for a *transitive* map f. Then, by [Ba97, Lemma 2.1 and Theorem 2.1],

(RPD3) $\overline{f^l(D_i)} = D_{i+l (\mathrm{mod}\, m)}$ for $0 \le i \le m-1$ and $l \ge 0$;
(RPD4) $f^{-l}(\mathrm{Int}(D_i)) \subseteq \mathrm{Int}\left(D_{i-l (\mathrm{mod}\, m)}\right)$ for $0 \le i \le m-1$ and $l \ge 0$;
(RPD5) $D = \bigcup_{i \ne j} D_i \cap D_j$ is closed, f-invariant and nowhere dense;
(RPD6) f^m is transitive on each D_i.

If all the sets of $\mathcal{D}$ are connected we say that $\mathcal{D}$ is *connected*. By (RPD3) and the fact that the continuous image of a connected set is connected we get that

(RPD7) if one of the sets D_i is connected then $\mathcal{D}$ is connected.

Lemma 15 ([Ba97], Corollary 2.1). *Let X be a topological space and f be a transitive map on X with f^n non-transitive for some $n \ge 2$. Then f has a regular periodic decomposition of length dividing n.*

Assume now that $\mathcal{D} = (D_0, \dots, D_{m-1})$ and $\mathcal{C} = (C_0, \dots, C_{n-1})$ are RPD's for f. We say that $\mathcal{C}$ *refines* (or is a *refinement* of) $\mathcal{D}$ if every C_i is contained in some D_j. Then each element of $\mathcal{D}$ contains the same number of elements of $\mathcal{C}$, so n is a multiple of m.

The following is a slight generalization of [Ba97, Theorem 6.1]; it will be used in the proof of Lemma 18.

Lemma 16. *Let X be a topological space containing a nonempty open locally connected subset. Let f be a transitive map on X. Then every regular periodic decomposition for f has a connected refinement.*

Proof. Let $J \ne \emptyset$ be an open locally connected subset of X and let $\mathcal{D} = \{D_0, \dots, D_{n-1}\}$ be a regular periodic decomposition for f. By (RPD2) the union of interiors of D_i is dense, so at least one of them — say $\mathrm{Int}\, D_0$ — intersects J. Let C_0 be a connected component of D_0 the interior of which intersects J (use that J is locally connected). By (RPD6), $f^n|_{D_0} : D_0 \to D_0$ is transitive. It follows, since C_0 is a component of D_0 with nonempty interior, that the set D_0 has finitely many components $C_0, C_1, \dots, C_{m-1}$ ($m \ge 1$) which are permuted by f^n and $\mathcal{C} = \{C_0, C_1, \dots, C_{m-1}\}$ is a regular periodic decomposition for $f^n|_{D_0}$. Let $G = \mathrm{Int}(C_0)$ (interior in X, not in D_0). By [Ba97, Lemma 3.3] we obtain that $\mathcal{E} = \{\overline{G}, \overline{f^{-(mn-1)}(G)}, \dots, \overline{f^{-(mn-2)}(G)}, \overline{f^{-1}(G)}\}$ is an RPD

for f. By (RPD4) it is a refinement of $\mathcal{D}$. Since $\overline{G} = C_0$ is connected, by (RPD7) we get that $\mathcal{E}$ is connected. $\square$

Given a transitive map f, by [Ba97] the set of all $m \in \mathbb{N}$ such that f admits an RPD of length m is called the *decomposition ideal* of f (the use of the term "ideal" is justified by the fact that this set is an ideal in the lattice of positive integers ordered by divisibility). If the decomposition ideal of f is finite then there is an RPD of f of maximal length. Such an RPD (which is by [Ba97, Theorem 2.2] unique up to cyclic permutations of its elements) is called a *terminal decomposition* of f.

Lemma 17 ([Ba97], Theorem 3.1)**.** *Let X be a topological space and f be a transitive map on X. Assume that $\mathcal{D} = (D_0, \dots, D_{m-1})$ is a regular periodic decomposition for f. Then $\mathcal{D}$ is terminal if and only if $f^m|_{D_i}$ is totally transitive for $0 \le i \le m-1$.*

Now we study regular periodic decompositions for transitive maps on spaces with a free interval.

Lemma 18. *Let X be a topological space with a free interval J and f be a transitive map on X. Assume that f has a periodic point x in J with period p. Then every regular periodic decomposition $\mathcal{D}$ for f has length at most $2p$. In particular, f has a terminal RPD.*

Proof. Fix an RPD $\mathcal{D} = (D_0, \dots, D_{m-1})$ for f. By Lemma 16 we may assume that $\mathcal{D}$ is connected. Two cases are possible:

(1) $x \in \mathrm{Int}(D_i)$ for some i;
(2) $x \in D_i \cap D_j$ for some $i \neq j$.

In the case (1), the periodicity of x and (RPD4) give
$$x \in f^{-p}(\mathrm{Int}(D_i)) \subseteq \mathrm{Int}(D_{i-p\,(\mathrm{mod}\,m)}).$$
Thus $\mathrm{Int}(D_i) \cap \mathrm{Int}(D_{i-p\,(\mathrm{mod}\,m)}) \neq \emptyset$ and so $m|p$ by (RPD1). Consequently $m \le p \le 2p$.

In the case (2), since $x \in J$ and D_i, D_j are connected, we may choose the ordering in J in such a way that there are $a \prec x \prec b$ in J with $(a, x] \subseteq D_i$ and $[x, b) \subseteq D_j$. Put $L = (a, x)$ and $R = (x, b)$. Since f is transitive neither $f^p(L)$ nor $f^p(R)$ is a singleton. However, $f^p(x) = x$ and so $f^p(z) \in (a, b)$ for all z sufficiently close to x. Hence
$$f^p(L) \cap (L \cup R) \neq \emptyset \qquad \text{and} \qquad f^p(R) \cap (L \cup R) \neq \emptyset. \tag{3.1}$$

If $f^p(L) \cap L \neq \emptyset$ then we have
$$\emptyset \neq \mathrm{Int}(D_i) \cap f^p(D_i) \subseteq \mathrm{Int}(D_i) \cap D_{i+p\,(\mathrm{mod}\,m)},$$
whence $m|p$ by (RPD1) and so $m \le p \le 2p$. The same inequality $m \le p$ is obtained if $f^p(R) \cap R \neq \emptyset$. It remains to consider the situation when $f^p(L) \cap L = \emptyset = f^p(R) \cap R$, $f^p(L) \cap R \neq \emptyset$ and $f^p(R) \cap L \neq \emptyset$. Hence, since $f^p(x) = x$, there are $a \prec a' \prec x \prec b' \prec b$ such that for the non-degenerate sets $f^p(L)$ and $f^p(R)$ we have $f^p(L) \supseteq R' := (x, b')$ and $f^p(R) \supseteq L' := (a', x)$. Since (3.1) obviously holds for L', R' instead of L, R we get that $f^{2p}(L) \cap L \neq \emptyset$. Analogously as we obtained above $m|p$ when $f^p(L) \cap L \neq \emptyset$ now we get $m|2p$ and so $m \le 2p$. $\square$

4. Periodic-recurrent property

In a system (X, f) always $\mathrm{Per}(f) \subseteq \mathrm{Rec}(f)$. The sets $\mathrm{Per}(f)$ and $\mathrm{Rec}(f)$ need not be closed. When $\overline{\mathrm{Per}(f)} = \overline{\mathrm{Rec}(f)}$ for every continuous map f on X, we speak on the *periodic-recurrent property* of the space X. Some one-dimensional spaces do have this property. In [Il00] dendrites with periodic-recurrent property have been characterized. For the history of the investigation of this property see [Il00] and [MS09]. A space with a disconnecting interval J need not be one-dimensional but the periodic-recurrent property, relatively in J, still holds.

Theorem 19. *Let X be a connected topological space with a disconnecting interval J and let $f : X \to X$ be a continuous map. Then*
$$\overline{\mathrm{Rec}(f)} \cap J = \overline{\mathrm{Per}(f)} \cap J.$$

Proof. Only one inclusion needs a proof. Moreover, it is sufficient to show that $\mathrm{Rec}(f) \cap J \subseteq \overline{\mathrm{Per}(f)} \cap J$ (it is elementary to check that then also $\overline{\mathrm{Rec}(f)} \cap J \subseteq \overline{\mathrm{Per}(f)} \cap J$). So fix a recurrent point $r \in J$ and consider any open interval J' with $r \in J' \subseteq J$. We show that $\mathrm{Per}(f) \cap J' \neq \emptyset$. If r itself is periodic then there is nothing to prove. So assume that $r \notin \mathrm{Per}(f)$. There are positive integers n, m such that $f^m(r), f^{m+n}(r) \in J'$ and either $r \prec f^{m+n}(r) \prec f^m(r)$ or $f^m(r) \prec f^{m+n}(r) \prec r$. Without loss of generality we may assume that the first possibility holds. We thus have $f^m(r) \succ r$ and $f^n(f^m(r)) \prec f^m(r)$. By Lemma 4, f has a periodic point in J'. ☐

If U is an open set in a topological space X and A is any set in X then $\overline{\overline{A} \cap U} = \overline{A \cap U}$. Therefore it follows from our theorem that

$$\overline{\mathrm{Rec}(f) \cap J} = \overline{\mathrm{Per}(f) \cap J}$$

which is the equality of two sets which are not necessarily subsets of J. One can see that both equalities are equivalent.

In general, Theorem 19 is no longer valid if J is assumed to be a free interval rather than a disconnecting one (consider an irrational rotation of the circle). One can deduce from [Bl86] that for graph maps a weaker form of the periodic-recurrent property holds, namely $\overline{\mathrm{Rec}(f)} = \mathrm{Rec}(f) \cup \overline{\mathrm{Per}(f)}$. Recently Mai and Shao gave in [MS09] a different proof of this fact. Their idea can be used to show that such an equality holds (relatively) in every free interval, see Theorem 20. Though the proof is pretty similar to that from [MS09], we include it into the appendix because the result is crucial for Lemma 21.

Theorem 20. *Let X be a topological space with a free interval J and let $f : X \to X$ be a continuous map. Then*

$$\overline{\mathrm{Rec}(f)} \cap J = \left[\mathrm{Rec}(f) \;\cup\; \overline{\mathrm{Per}(f)}\right] \cap J.$$

5. Density of (eventually) periodic points

From now on we consider only compact metrizable spaces. The reason is that we use results known only in such spaces (results on cantoroids) and results which do not work without compactness (Lemma 13 and the fact that transitive maps in compact metrizable spaces have dense set of transitive points).

The following two lemmas are first steps towards the proof of the dichotomy for transitive maps, see Theorem C.

Lemma 21. *Let X be a compact metrizable space with a free arc A and let $f : X \to X$ be a transitive map. Assume that f has a periodic point x_0 in A. Then the set of periodic points of f is dense in X.*

Proof. For a transitive map, the set of periodic points is either nowhere dense or dense. So it is sufficient to show that the periodic points of f are dense in A. Assume, on the contrary, that there is a free interval $J \subseteq A$ with $J \cap \mathrm{Per}(f) = \emptyset$. The space X has no isolated point (otherwise, due to transitivity of f, it would be finite). By Lemma 13, the set $\mathrm{Tr}(f)$ of transitive points of f is dense in X and hence dense in J. Every transitive point is clearly recurrent, so $\mathrm{Rec}(f)$ is dense in J. By Theorem 20 we have

$$J = \overline{\mathrm{Rec}(f)} \cap J = \left[\mathrm{Rec}(f) \cup \overline{\mathrm{Per}(f)}\right] \cap J = \mathrm{Rec}(f) \cap J.$$

Thus every point of J is recurrent and hence no point of J is eventually mapped to x_0.

For $n \geq 0$ put $J_n = f^n(J)$ and consider the set $Y = \bigcup_{n=0}^{\infty} J_n$. Then $x_0 \notin Y$. Further, Y is f-invariant and is dense in X by transitivity of f. Therefore the restriction of f to Y is also transitive. Since Y contains a nonempty open connected set J, Y has only finitely many connected components, say $Y_0, \ldots, Y_{p-1}$, they are cyclically permuted by f and the restriction of f^p to each of them is topologically transitive. We may assume that Y_0 is the component of Y containing J. So Y_0 is a connected space with a free interval J.

We claim that J is in fact a disconnecting interval for Y_0. According to Proposition 44(i) it is sufficient to find a point $x \in J$ such that $Y_0 \setminus \{x\}$ is not connected. We show that every $x \in J$ works. To this end fix $x \in J$. Then $x \neq x_0$ and we may assume that $x \prec x_0$. Notice that, since X is Hausdorff, the compact set $[x, x_0]$ is closed in X, hence $X \setminus [x, x_0]$ is open. Then (x, x_0) and $X \setminus [x, x_0]$ are disjoint open sets in X whose union contains $Y_0 \setminus \{x\}$. It follows easily that $(x, x_0) \cap Y_0$ and $Y_0 \setminus [x, x_0]$ form a separation of $Y_0 \setminus \{x\}$, so $Y_0 \setminus \{x\}$ is not connected.

By Lemma 5 the periodic points of $f^p|_{Y_0}$ are dense in Y_0. Consequently, the periodic points of f are dense in J which is a contradiction. $\square$

Lemma 22. *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a transitive map. Assume that there is a nonempty closed nowhere dense invariant set $S \subseteq X$ such that $S \cap J \neq \emptyset$. Then the set of periodic points of f is dense in X.*

Proof. Let π denote the quotient map which collapses the set S into a point, call it s, and denote the corresponding quotient space of X by Y. Obviously, the underlying decomposition of X is upper semi-continuous which implies that Y is compact and metrizable. Since S is invariant for f, we have an induced dynamics on Y. More precisely, there is a continuous map $g : Y \to Y$ with $g \circ \pi = \pi \circ f$. The restriction of π to $X \setminus S$ is a homeomorphism onto $Y \setminus \{s\}$ (it is a continuous bijection and it is open since every open subset of $X \setminus S$ is saturated). So, since S is nowhere dense, to prove the density of $\mathrm{Per}(f)$ in X it is sufficient to show that $\mathrm{Per}(g)$ is dense in Y.

Choose a free arc A in $J \subseteq X$ whose intersection with S is just one point, an end point of A. Then $B = \pi(A)$ is a free arc in Y whose one end point is s. Further, (Y, g) is a transitive system, being a factor of (X, f). Finally, $s \in B$ is a fixed point of g. By Lemma 21 applied to g we obtain that $\mathrm{Per}(g)$ is dense in Y. $\square$

If f is a transitive map on an infinite compact metrizable space with dense periodic points then it can happen that there are no points in the system which are eventually periodic but not periodic. Such an example can be found in [DY02, Section 3]. (It is a so-called ToP-system, see also the end of Section 7). The following lemma shows that under the additional assumption that the space has a free interval the eventually periodic points do exist.

Lemma 23. *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a transitive map with dense set of periodic points. Then the set of all eventually periodic points of f which are not periodic is dense in X.*

Proof. By Lemma 18, f has a terminal regular periodic decomposition $\mathcal{D} = (D_0, \dots, D_{m-1})$. There is $i \in \{0, \dots, m-1\}$ such that $\mathrm{Int}(D_i) \cap J \neq \emptyset$. Then D_i is a compact metrizable space with a free interval J' (a subinterval of J). Moreover, by Lemma 17, the map $g = f^m|_{D_i}$ is totally transitive on D_i. Since D_i is regular closed and $\mathrm{Per}(f)$ is dense in X we get that $\mathrm{Per}(g)$ is dense in D_i. By Lemma 3, g is weakly mixing. Now choose a periodic point p of g satisfying $\mathrm{card}(\mathrm{Orb}_g(p) \cap J') \geq 3$. Let $p_1 \prec p_2 \prec p_3$ be members of $\mathrm{Orb}_g(p)$ lying in J'. We show that the set of points non-periodic for g which are eventually mapped into $\mathrm{Orb}_g(p)$ (and hence are eventually periodic for g) is dense in J'. To this end we prove that any open interval $V \subseteq J' \setminus \mathrm{Orb}_g(p)$ contains such a point. Since g is weakly mixing, there exists $k \in \mathbb{N}$ with $g^k(V) \cap (p_j, p_{j+1}) \neq \emptyset$ for $j = 1, 2$. Then the set $g^k(V)$, being connected, contains p_j for some $j \in \{1, 2, 3\}$. Thus there exists a point $e \in V$ which is mapped by g^k into $\mathrm{Orb}_g(p)$. Since $e \notin \mathrm{Orb}_g(p)$, e is eventually periodic but not periodic for g.

We have shown that points in J' which are eventually periodic but not periodic for g are dense in J'. Trivially, in this statement we may replace g by f.

To finish the proof fix a nonempty open set U in X. By transitivity of f there is an open subinterval K of J' and integers $0 \leq j \leq n$ such that $f^j(K) \subseteq U$ and $f^n(K) \subseteq J'$. The interval $f^n(K)$ is non-degenerate and so it contains a point y_0 which is eventually periodic but not periodic for f. Since $f^{(n-j)}(U) \supseteq f^n(K)$, there is $x_0 \in U$ with $f^{(n-j)}(x_0) = y_0$. The point x_0 is eventually periodic but not periodic for f which proves the density of such points in X. $\square$

6. Minimality and a trichotomy for minimal sets (proofs of Theorems A and B)

We embark on the proof of Theorems A and B.

Lemma 24. *Let X be a second countable Hausdorff space and J be a nonempty subset of X. Then J is a free interval if and only if it is open, connected, locally homeomorphic to $\mathbb{R}$ and it is not a circle.*

Proof. One implication is trivial, for the converse one use the fact that if J is connected and locally homeomorphic to $\mathbb{R}$ then, being a connected one-dimensional topological manifold, it is either a circle or an open interval. □

Notice that if J is a free interval in a circle X then always there is maximal (with respect to the inclusion) free interval J^* containing J (in fact, J^* is the circle minus a point). However, if the complement of J is not a singleton then J^* is not unique. If X is not a circle, the following lemma shows that the things work better.

Lemma 25. *Let X be a continuum which is not a circle.*

(a) *If J_1, J_2 are free intervals in X then either they are disjoint or their union is again a free interval.*
(b) *Two maximal (with respect to the inclusion) free intervals of X either are disjoint or coincide.*
(c) *If J is a free interval in X then it is a subset of a (unique) maximal free interval.*

Proof. (a) Assume that $J_1 \cap J_2 \neq \emptyset$. Then $J = J_1 \cup J_2$ is a connected open subset of X. Suppose that J is a circle. Then J is closed in X and so it is a clopen subset of X. Hence $X = J$ by connectedness of X, a contradiction. In view of Lemma 24 it remains to show that J is locally homeomorphic to $\mathbb{R}$. Fix $x \in J$; we may assume that $x \in J_1$. Since J_1 is a free interval there is a neighborhood U of x (in the topology of X) which is homeomorphic to $\mathbb{R}$ and is a subset of J_1, hence a subset of J. So, U is a neighborhood of x (in the topology of J) homeomorphic to $\mathbb{R}$.

(b) Indeed, if J_1 and J_2 are maximal free intervals in X with $J_1 \cap J_2 \neq \emptyset$ then, by (a), $J_1 \cup J_2$ is a free interval in X. By maximality of both J_1 and J_2 we have $J_1 = J_1 \cup J_2 = J_2$.

(c) Let J^* denote the union of all free intervals containing J. Then, analogously as in (a), J^* is a free interval. Obviously it is a maximal free interval containing J. Uniqueness follows from (b). □

Lemma 26. *Let X be a continuum with a free interval and let f be a transitive homeomorphism on X. Then X is a circle and f is conjugate to an irrational rotation.*

Proof. If X is a circle use Lemma 14. Supposing that X is not a circle, we are going to find a space Y homeomorphic to a circle and a transitive homeomorphism g on it with a fixed point, which will contradict Lemma 14. The system (Y, g) will be obtained as a factor of (X, f^n) for some $n > 0$.

By Lemma 25(c) there is a maximal free interval and since f is a homeomorphism, every maximal free interval is mapped onto such an interval. Then transitivity of f, in view of Lemma 25(b), gives that there are only finitely many pairwise disjoint maximal free intervals, say $J_1, \dots, J_n$ ($n \geq 1$), and f permutes them in a periodic way. Since X is compact and the free intervals $J_1, \dots, J_n$ are pairwise disjoint, the set $X \setminus \bigcup_{i=1}^n J_i$ is nonempty. It is closed, f-invariant and, by transitivity of f, nowhere dense.

Denote by $\mathcal{D}$ the decomposition of X whose elements are the singletons $\{x\}$ with $x \in J_1$ and the (closed) set $X \setminus J_1$. Obviously $\mathcal{D}$ is upper semi-continuous and so the decomposition space $Y = X/\mathcal{D}$ is a (metrizable) continuum. Denote by π the quotient map $X \to Y$. Clearly, $\pi(J_1)$ is a free interval in Y. Since $Y \setminus \pi(J_1)$ is a singleton, the space Y, being a one-point compactification of an open interval, is a circle.

The map f^n is a homeomorphism of X and both J_1 and $X \setminus J_1$ are f^n- invariant. Consequently, there exists a homeomorphism g of Y with $g \circ \pi = \pi \circ f^n$. Moreover, an elementary argument gives that $f^n|_{J_1}$ is transitive (alternatively one can use (RPD6) for the regular periodic decomposition $(J_1, \dots, J_n)$ for f restricted to $\bigcup_i J_i$). Therefore also $g|_{\pi(J_1)}$, being conjugate to $f^n|_{J_1}$, is transitive.

It follows that the homeomorphism g is transitive on the circle Y. However, the singleton $\pi(X \setminus J_1)$ is a fixed point for g which contradicts Lemma 14. ☐

Corollary 27. *A continuum with a disconnecting interval does not admit a transitive homeomorphism.*

Proof. Consider an appropriate iterate of the homeomorphism and apply Lemma 26. ☐

Of course in this corollary it is substantial that we speak on homeomorphisms (the tent map is a transitive map on a continuum with a disconnecting interval).

Proposition 28. *Let X be a continuum with a free interval J and let f be a minimal map on X. Then X is a circle and f is conjugate to an irrational rotation.*

Proof. In view of Lemma 26 it is sufficient to show that f is one-to-one. We proceed by contradiction. Suppose that $f(x) = f(y)$ for some $x \neq y$. By Lemma 11 there are $n \in \mathbb{N}$ and $a \in J$ with $f^n(a) = x$. Now use surjectivity of f to find $b \in X$ with $f^n(b) = y$. There is $k \geq n+1$ such that $c = f^k(a) = f^k(b) \in J$. Notice that a, b, c are pairwise distinct (because $x \neq y$ and f has no periodic point) and $a, c \in J$. Moreover, by Lemma 12, f^k is minimal.

Before proceeding further realize that the following claim holds.

Claim. Let Y be a topological space and $g : Y \to \mathbb{R}$ be a continuous map. Let U, V be two disjoint nonempty open sets in Y such that $g(U)$ and $g(V)$ overlap (i.e. there is an open interval L with $L \subseteq g(U) \cap g(V)$). Then there is a nonempty open set G in Y with $g(G) \subseteq g(Y \setminus G)$.

The proof of the claim is obvious, just set $G = U \cap g^{-1}(L)$.

To finish the proof of the proposition choose an open interval $a \in A \subseteq J$ and an open neighborhood $B \ni b$ such that A, B are disjoint and both $f^k(A)$ and $f^k(B)$ are subsets of J. The set $A \setminus \{a\}$ consists of two disjoint open intervals A_1, A_2 with a being their common limit point.

Both $f^k(A_1)$ and $f^k(A_2)$ contain an open interval having the point c as its left or right end point. Moreover, by Lemma 8, we have $\operatorname{Int} f^k(B') \neq \emptyset$ for every neighborhood $B' \subseteq B$ of b. It follows that either the sets $f^k(A_1), f^k(A_2)$ overlap or one of them overlaps with $f^k(B)$. In any case we may use the claim, with $Y = A \cup B$ and $g = f^k|_Y$, to find an open redundant set for f^k which contradicts the minimality of f^k, see Lemma 7. ☐

Now we are ready to prove Theorems A and B. For reader's convenience we repeat the statements here.

Theorem A (**Minimal spaces with a free interval**)**.** *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a minimal map. Then X is a disjoint union of finitely many circles, $X = \bigoplus_{i=0}^{n-1} \mathbb{S}^1_i$, which are cyclically permuted by f and, on each of them, f^n is topologically conjugate to the same irrational rotation.*

Proof. Since f is minimal and X has a component with nonempty interior (namely the one containing J), X has only finitely many components $C_0, \dots, C_{n-1}$ and they are cyclically permuted by f. We may assume that $C_0 \supseteq J$ and $f(C_i) = C_{i+1 (\mathrm{mod}\, n)}$ for all $0 \leq i \leq n-1$. Since $f^n|_{C_0}$ is minimal and C_0 is a continuum with a free interval J, we have, by Proposition 28, that C_0 is a circle and $f^n|_{C_0}$ is conjugate to an irrational rotation. Now fix $i \in \{1, \dots, n-1\}$. Since $f^n : C_0 \to C_0$ is a homeomorphism, it follows that also $f^i : C_0 \to C_i$ is a homeomorphism and hence it is a conjugacy between $f^n|_{C_0}$ and $f^n|_{C_i}$. So all the maps $f^n|_{C_i}$ $(i = 0, \dots, n-1)$ are conjugate to the same irrational rotation which completes the proof. ☐

Theorem B (**Trichotomy for minimal sets**)**.** *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a continuous map. Assume that M is a minimal set for f which intersects J. Then exactly one of the following three statements holds.*

(1) *M is finite.*
(2) *M is a nowhere dense cantoroid.*

(3) *M is a disjoint union of finitely many circles.*

Proof. Trivially the three cases are mutually exclusive. Assume first that $M \cap J$ contains an arc. Then M is a compact metrizable space with a free interval admitting a minimal map $f|_M$. By Theorem A, M is a disjoint union of finitely many circles and so we have (3).

Now we assume that $M \cap J$ contains no arc and we show that in this case either (1) or (2) holds. Clearly, $M \cap J$ is totally disconnected. If $M \cap J$ has an isolated point then M has an isolated point and so M, being a minimal set, is finite. So assume that $M \cap J$ is dense in itself. We prove (2). The set M, being an infinite minimal set, has no isolated point. Thus to show that M is a cantoroid we only need to prove that the union of all degenerate components of M is dense in M. By Lemma 9, the set R of all points $x \in M$ which have a unique preimage in M is residual in M. By Lemma 10 all the images $f^n(R)$ of R are residual in M. Since $M \cap J$ is a nonempty open subset of M it necessarily contains a point z_0 from the residual (in M) set $\bigcap_{n=0}^{\infty} f^n(R)$. Since $z_0 \in M \cap J$, it is a degenerate component of M and since $z_0 \in \bigcap_{n=0}^{\infty} f^n(R)$, it has a unique backward orbit $\{z_n\}_{n=0}^{\infty}$ with respect to $f|_M$. By Lemma 11, this orbit is dense in M. Since z_0 is a component of M, the singleton $z_n = (f|_M)^{-n}(z_0)$, being a union of components of M, is itself a component of M. Hence degenerate components of M are dense in M and so M is a cantoroid.

It remains to show that M is nowhere dense in X. Suppose, on the contrary, that the (closed) set M contains a set U which is nonempty and open in X. Fix an arc $A \subseteq J$ containing a nonempty open subset of M (in the topology of M). Since M is minimal for f, there is $n \in \mathbb{N}$ such that $\bigcup_{i=0}^{n} f^i(A) \supseteq M \supseteq U$. Since the sets $f^i(A)$ are closed there is j such that $f^j(A) \supseteq V$ for some nonempty open set $V \subseteq U$. The set $f^j(A)$, being a continuous image of an arc, is locally connected. Therefore we may assume that V is connected. Since M is minimal and intersects J, there is $k \in \mathbb{N}$ with $f^k(V) \cap J \neq \emptyset$. Since the components of $M \cap J$ are singletons, it follows that $f^k(V)$ is also a singleton. This implies that M is finite, contradicting our assumptions. □

7. Transitivity and dense periodicity

Theorem C will follow from Lemma 26 and from three other lemmas, each of which will deal with some 'partial' dichotomy for transitive maps (we call them partial because Theorem C called a "Dichotomy for transitive maps" combine all of them). In this section we prove the first of these three lemmas.

Lemma 29. *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a transitive map. Then exactly one of the following two statements holds.*

(1) *The periodic points of f form a dense subset of X.*
(2) *The space X is a disjoint union of finitely many circles, $X = \bigoplus_{i=0}^{n-1} \mathbb{S}_i^1$, which are cyclically permuted by f and, on each of them, f^n is topologically conjugate to the same irrational rotation.*

Proof. By Lemma 13 the set $\mathrm{Intr}(f)$ of intransitive points of f is either empty or dense. If it is empty, then (X, f) is minimal and Theorem A gives (2).

So assume that the set $\mathrm{Intr}(f)$ is dense in X and fix $x_0 \in J \cap \mathrm{Intr}(f)$. Denote by S the closure of the orbit of x_0 under f. Clearly, S is a nonempty closed invariant set. Furthermore, we have $S \subseteq \mathrm{Intr}(f)$. Indeed, if S contained a transitive point y_0 then, being invariant, it would contain the orbit of y_0 and hence it would be dense in X, contradicting the fact that x_0 is not transitive. So $S \subseteq \mathrm{Intr}(f)$ and, since $\mathrm{Tr}(f)$ is dense in X, the (closed) set S is nowhere dense in X. Lemma 22 then gives (1). □

By [DY02], a dynamical system (X, f) is called a *ToM-system* if X is a compact metrizable space, f is transitive, not minimal, and every point of X is either transitive or minimal (a point is called *minimal* if it belongs to a minimal set). If in such a system every point is either transitive or periodic, the system is called a *ToP-system*. Notice that if (X, f) is a ToM-system then X has no isolated point. (Otherwise, by transitivity of f, the system would be just one periodic orbit, so it would be minimal.)

Corollary 30. *Let X be a compact metrizable space with a free interval J. Then there are no ToM-systems on X.*

Proof. Suppose that (X, f) is a ToM-system. Since f is transitive and not minimal, by Lemma 29 it has a dense set of periodic points. By Lemma 23, f has a dense set of points which are eventually periodic but not periodic, a contradiction. ☐

8. Transitivity and topological entropy

The following result should be well known but we have not found it explicitly and so we include a proof.

Lemma 31. *Let X be a compact metrizable space and $f : X \to X$ a continuous map. Assume that there exist nonempty closed pairwise disjoint subsets $A_1, \dots, A_m$ of X and a positive integer k such that for every $i \in \{1, \dots, m\}$ there is $p_i \in \mathbb{N}$ with*

$$\operatorname{card}\{j \in \{1, \dots, m\} : f^{p_i}(A_i) \supseteq A_j\} \geq k.$$

Then for $p = \max\{p_1, \dots, p_m\}$ we have $h(f) \geq (1/p) \log k$.

Proof. Call a finite sequence $\overline{s} = (s_0, \dots, s_n)$ $(n \geq 0)$ of elements of the set $\{1, \dots, m\}$ realizable if $f^{p_{s_i}}(A_{s_i}) \supseteq A_{s_{i+1}}$ for $0 \leq i \leq n-1$. For such a sequence $\overline{s}$ the set

$$A_{\overline{s}} = \{x \in A_{s_0} : \ f^{p_{s_0}}(x) \in A_{s_1}, f^{p_{s_0}+p_{s_1}}(x) \in A_{s_2}, \dots, f^{p_{s_0}+\dots+p_{s_{n-1}}}(x) \in A_{s_n}\}$$

is obviously nonempty and so one can fix $x_{\overline{s}} \in A_{\overline{s}}$. For $n \in \mathbb{N}$ put

$$E_n = \{x_{\overline{s}} : \ \overline{s} = (s_0, \dots, s_n) \text{ is realizable}\}.$$

By our assumptions $\operatorname{card} E_n \geq mk^n \geq k^{n+1}$. Choose $0 < \varepsilon < \min_{i \neq j} \operatorname{dist}(A_i, A_j)$. It follows that the set E_n is $(np+1, \varepsilon)$-separated for f. Consequently, by Bowen's definition of topological entropy,

$$h(f) \geq \limsup_{n\to\infty} \frac{1}{np+1} \log(\operatorname{card} E_n) \geq \limsup_{n\to\infty} \frac{n+1}{np+1} \log k = \frac{1}{p} \log k,$$

as desired. ☐

Lemma 32. *Let X be a compact metrizable space with a free interval J. Then every weakly mixing map f on X has positive topological entropy.*

Proof. We prove the following assertion:

(A) If a connected set $C \subseteq X$ contains three distinct points $x_1, x_2, x_3 \in J$, then it contains at least one of the (non-degenerate) subarcs of J whose end points are in $\{x_1, x_2, x_3\}$.

We may assume that $x_1 \prec x_2 \prec x_3$. Suppose that C contains neither $[x_1, x_2]$ nor $[x_2, x_3]$. Fix $y_1 \in [x_1, x_2] \setminus C$ and $y_2 \in [x_2, x_3] \setminus C$. Then the sets (y_1, y_2) and $X \setminus [y_1, y_2]$ constitute a separation of C in X which is a contradiction.

Now we prove the lemma. Fix three disjoint arcs A_1, A_2, A_3 in J. Then the set $J \setminus \bigcup_{i=1}^3 A_i$ has four components U_j $(1 \leq j \leq 4)$ which are open in X. Since f is weakly mixing, there is $p \in \mathbb{N}$ satisfying $f^p(A_i) \cap U_j \neq \emptyset$ for every i, j. The Assertion (A) now secures that for every i the connected set $f^p(A_i)$ contains at least two of the sets A_1, A_2, A_3. Thus, by Lemma 31, $h(f) \geq (1/p) \log 2 > 0$. ☐

We get the second partial dichotomy for transitive maps.

Lemma 33. *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a transitive map. Then exactly one of the following two statements holds.*

1. *The topological entropy of f is positive.*
2. *The space X is a disjoint union of finitely many circles, $X = \bigoplus_{i=0}^{n-1} \mathbb{S}^1_i$, which are cyclically permuted by f and, on each of them, f^n is topologically conjugate to the same irrational rotation.*

Proof. Assume that f does not satisfy condition (2). Then, by Lemma 29, the set of periodic points of f is dense in X. By Lemma 18 there is a terminal regular periodic decomposition $\mathcal{D} = (D_0, \dots, D_{m-1})$ for f. We may assume that D_0 intersects J and so D_0 is compact metrizable with a free interval. By Lemma 17 the map $f^m|_{D_0}$ is totally transitive. Since the periodic points of f are dense in X and D_0 is regular closed in X, it follows that the map $f^m|_{D_0}$ has dense periodic points and, by Lemma 3, is weakly mixing. It follows from Lemma 32 that $h(f^m|_{D_0}) > 0$ and so $h(f) > 0$. $\square$

9. Transitivity and strong mixing. Proof of Theorem C

To prove Theorem C we will need the following, already the third, partial dichotomy for transitive maps. (Recall that f is relatively strongly mixing if f has an RPD $(D_0, \dots, D_{m-1})$ such that $f^m|_{D_i}$ is strongly mixing for every i.)

Lemma 34. *Let X be a compact metrizable space with a free interval J and let $f : X \to X$ be a transitive map. Then exactly one of the following two statements holds.*

1. *The map f is relatively strongly mixing.*
2. *The space X is a disjoint union of finitely many circles, $X = \bigoplus_{i=0}^{n-1} \mathbb{S}^1_i$, which are cyclically permuted by f and, on each of them, f^n is topologically conjugate to the same irrational rotation.*

Proof. The proof is word by word the same as that of Lemma 33 only at the very end of it we use Lemma 6 instead of Lemma 32, to get that $f^m|_{D_0}$ is strongly mixing. Then every $f^m|_{D_i}$, being a factor of $f^m|_{D_0}$ by (RPD3), is also strongly mixing. $\square$

Theorem C (Dichotomy for transitive maps). *Let X be a compact metrizable space with a free interval and let $f : X \to X$ be a transitive map. Then exactly one of the following two statements holds.*

1. *The map f is relatively strongly mixing, non-invertible, has positive topological entropy and dense periodic points.*
2. *The space X is a disjoint union of finitely many circles, $X = \bigoplus_{i=0}^{n-1} \mathbb{S}^1_i$, which are cyclically permuted by f and, on each of them, f^n is topologically conjugate to the same irrational rotation.*

Proof. Assume that (2) does not hold. Then f is relatively strongly mixing by Lemma 34, has positive entropy by Lemma 33 and has dense periodic points by Lemma 29. Finally, f is non-invertible by Lemma 26. $\square$

Appendix 1

As we promised in Section 4, we prove here Theorem 20. For maps f and g, each of them being defined on a (possibly different) subset of X and with values in X, put

$$\mathrm{P}_{g,f} = \mathrm{Fix}(f) \cup \mathrm{Fix}(g) \cup \bigcup_{k \geq 1} \mathrm{Fix}(g^k \circ f).$$

For an interval map φ we will write $[\alpha, \beta] \overset{\varphi}{\searrow} [\gamma, \delta]$ if $\varphi([\alpha, \beta]) = [\gamma, \delta]$, $\varphi(\alpha) = \delta$ and $\varphi(\beta) = \gamma$. In the proof of the next lemma we will use the following simple observation: If $\varphi([\alpha', \beta']) \supseteq [\gamma, \delta]$, $\varphi(\alpha') = \delta$ and $\varphi(\beta') = \gamma$ then there are $\alpha' \leq \alpha < \beta \leq \beta'$ with $[\alpha, \beta] \overset{\varphi}{\searrow} [\gamma, \delta]$.

Lemma 35. *Let $a < d$ and $b, c \in (a, d)$ be reals and let $f : [a, b] \to [a, d]$ and $g : [c, d] \to [a, d]$ be interval maps such that $f(a) = d$, $g(c) = a$ and $g(d) \in f([a, b])$. Then $\mathrm{P}_{g,f} \neq \emptyset$.*

Proof. Assume that neither f nor g has a fixed point. Then $g(d) < d$ and, by the assumption, $\min_{x \in [a,b]} f(x) \leq g(d)$. By replacing b by appropriate $a < b' < b$ (and replacing f by $f' = f|_{[a,b']}$), if necessary, we may assume that $f(x) > f(b) = g(d)$ for every $x \in [a, b)$. Analogously by replacing

c by a larger number, if necessary, we may assume that $a = g(c) < g(y)$ for every $y \in (c, d]$. See Figure 1 (the choice $b < c$ in the figure does not play any role in the proof).

Since f and g have no fixed points, it holds

$$f(x) > x \quad \text{and} \quad g(y) < y \qquad \text{for every } x \in [a, b] \text{ and } y \in [c, d].$$

Thus $\min_{y\in[c,d]}(y - g(y)) > 0$ and hence there is $m \geq 1$ such that

$$d_0 := d \ > \ d_1 := g(d_0) \ > \ \dots \ > \ d_m := g(d_{m-1}) \ \geq \ c \ \ > \ d_{m+1} := g(d_m).$$

If $d_{m+1} = a$ then $a \in \mathrm{Fix}(g^{m+1} \circ f)$ and the proof is finished. So assume that $a < d_{m+1}$.

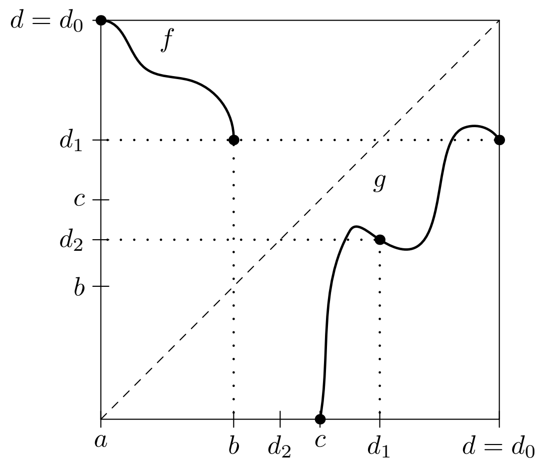

FIGURE 1. Illustration for the case when $m = 1$.

For $i \geq 0$ put $g_i = g^i \circ f$. Since $f([a, b]) \supseteq [d_1, d_0]$ and $g([d_i, d_{i-1}]) \supseteq [d_{i+1}, d_i]$ for $i = 1, \dots, m$, we get

$$g_i([a, b]) \supseteq [d_{i+1}, d_i], \qquad \text{for } i = 0, \dots, m$$

(of course, $g_i([a, b])$ means the g_i-image of the intersection of $[a, b]$ with the domain of g_i). So, by the observation above the lemma, one can find a_i, b_i such that

$$a = a_0 \leq a_1 \leq \dots \leq a_m \ < \ b_m \leq \dots \leq b_1 \leq b_0 = b \qquad \text{and} \qquad [a_i, b_i] \overset{g_i}{\searrow} [d_{i+1}, d_i].$$

Moreover, $g([d_{m+1}, d_m]) = g([c, d_m]) \supseteq [a, d_{m+1}]$ and so there are $a_{m+1} < b_{m+1}$ in $[a_m, b_m]$ such that

$$[a_{m+1}, b_{m+1}] \overset{g_{m+1}}{\searrow} [a, d_{m+1}].$$

Then $g_m(b_{m+1}) = c$ since $g^{-1}(a) = \{c\}$.

Put

$$p = \max\{0 \leq i \leq m+1 : \ b_i < g_i(b_i)\}.$$

Since $b < f(b)$ can be written as $b_0 < g_0(b_0)$, $p \geq 0$ exists. Moreover, $p \leq m$ since $b_{m+1} > a = g_{m+1}(b_{m+1})$. Now $g_{p+1}(b_{p+1}) \leq b_{p+1}$ (by the choice of p) and, regardless of whether $p \leq m-1$ or $p = m$, we get $g_{p+1}(a_{p+1}) = d_{p+1} = g_p(b_p) > b_p > a_{p+1}$. So $g_{p+1} = g^{p+1} \circ f$ has a fixed point in $[a_{p+1}, b_{p+1}]$. Hence $\mathrm{P}_{g,f} \neq \emptyset$. □

To simplify the things, in the following two lemmas we will assume that a given free arc A is (not only homeomorphic to $[0, 1]$ but) exactly $[0, 1]$. This of course does not restrict generality and enables us to imagine a part of the considered dynamics as an interval one.

Lemma 36. *Let X be a compact metrizable space with a free arc $A = [0, 1]$. Let $f : A \to X$ and $g : X \to X$ be continuous maps such that $A \cap \mathrm{P}_{g,f} = \emptyset$ and there are $0 < x < y < 1$ with $f(x), g(y) \in [x, y]$. Then the following are true:*

(1) *If* $f(x) \le g(y)$ *then* $f(0) \in (0,1)$.
(2) *If* $f(x) > g(y)$ *then* $f(0) \in (0,1)$ *or* $g(f(0)) \in (0,1)$.

Proof. (1) Assume, on the contrary, that $f(0) \notin (0,1)$. Let $v \in [0,x)$ be maximal such that $f(v) \in \{0,1\}$; such v exists since $f([0,x])$ is a connected set intersecting $(0,1)$ and $X \setminus (0,1)$. By maximality of v we get

$$f([v,x]) \subseteq [0,1].$$

Then $f(v)=1$ since otherwise $[v,x] \subseteq f([v,x]) \subseteq [0,1]$ and f would have a fixed point in $[v,x]$. Let $a \in [v,x]$ be maximal such that $f(a)=y$; then $f([a,x]) \subseteq [a,y]$. Finally let $c \in [a,y]$ be maximal such that $g(c)=a$ (it must exist since g has no fixed point in $[a,y]$ and $g(y)<y$); then $g([c,y]) \subseteq [a,y]$. Denote $f' = f|_{[a,x]}$ and $g' = g|_{[c,y]}$. Since $f(x) \le g(y) < y = f(a)$ we get $g'(y) \in f'([a,x])$. Now Lemma 35 applied to f', g' gives that $\mathrm{P}_{g',f'} \neq \emptyset$, a contradiction with $\mathrm{P}_{g',f'} \subseteq \mathrm{P}_{g,f} \cap A = \emptyset$.

(2) Now assume that $x \le g(y) < f(x) \le y$ and $f(0) \notin (0,1)$. As in the proof of (1) let $v \in [0,x)$ be maximal such that $f(v) \in \{0,1\}$. Then again $f(v)=1$ and $[f(x),1] \subseteq f([v,x]) \subseteq [0,1]$. So there is $x' \in (v,x)$ with $f(x')=y$. Put $f' = g \circ f$. Then $f' : A \to X$, $g : X \to X$, $x' < x \le g(y) \le y$ and $f'(x') = g(y)$. Moreover, $\mathrm{P}_{g,f'} \cap A \subseteq \mathrm{P}_{g,f} \cap A = \emptyset$. The already proved case (1) applied to f', g and x', y gives that $g(f(0)) = f'(0) \in (0,1)$. $\square$

Lemma 37. *Let* X *be a compact metrizable space with a free arc* $A=[0,1]$. *Let* $f : X \to X$ *be a continuous map such that* $\mathrm{Per}(f) \cap [0,1] = \emptyset$ *and* $\mathrm{Rec}(f) \cap (0,1) \neq \emptyset$. *Then* $\mathrm{Orb}_f(0) \cap (0,1) \neq \emptyset$.

Proof. Let $x \in \mathrm{Rec}(f) \cap (0,1)$; then there are positive integers m,n such that $f^n(x) \in (0,1)$ and $f^{n+m}(x)$ is between x and $f^n(x)$.

Assume first that $x < f^{n+m}(x) < f^n(x)$. We are going to apply Lemma 36(2) to the points $x, y = f^n(x)$ and the maps $F = f^n|_A, G = f^m$. Since trivially $\mathrm{P}_{G,F} \subseteq \mathrm{Per}(f)$, we have $\mathrm{P}_{G,F} \cap [0,1] = \emptyset$. Further, $0 < x < G(y) < F(x) = y < 1$. So Lemma 36(2) gives $F(0) \in (0,1)$ or $G(F(0)) \in (0,1)$. So $\mathrm{Orb}_f(0) \cap (0,1) \neq \emptyset$.

Now assume that $f^n(x) < f^{n+m}(x) < x$. Put $x' = f^n(x)$, $y' = x$ and $F = f^m|_A$, $G = f^n$. Then $0 < x' = G(y') < F(x') < y' < 1$. Again $\mathrm{P}_{G,F} \cap [0,1] = \emptyset$ and we can use Lemma 36(2) as in the previous case. $\square$

Proof of Theorem 20. We only need to prove the inclusion

$$\overline{\mathrm{Rec}(f)} \cap J \subseteq \left[\mathrm{Rec}(f) \ \cup \ \overline{\mathrm{Per}(f)}\right] \cap J.$$

Assume that $x \in J$ is such that $x \in \overline{\mathrm{Rec}}(f) \setminus \overline{\mathrm{Per}}(f)$. If x is recurrent we are done. So assume that $x \notin \mathrm{Rec}(f)$. Take arbitrarily small free interval $L \subseteq J$ containing x which is disjoint with $\mathrm{Per}(f)$. Choose an orientation on L which ensures that for some free arc $A = [x,b] \subseteq L$ there is a recurrent point of f in (x,b). By Lemma 37, the orbit of x intersects $(x,b) \subseteq L$. Since L was arbitrarily small, x is recurrent. This contradicts our assumption. $\square$

Appendix 2

We are going to show that in compact connected Hausdorff spaces, many natural definitions of a disconnecting interval are equivalent. However, in more general spaces this is not the case, see Proposition 44.

Let X be a topological space and J be a free interval. Recall that we assume that one of the two natural orderings on J (induced by the usual orderings on a real interval) is chosen and denoted by $\prec$. If K is a subinterval of J we define the (possibly empty) sets

$$J_K^- := \{z \in J : \ z \prec y \text{ for all } y \in K\} \qquad \text{and} \qquad J_K^+ := \{z \in J : \ y \prec z \text{ for all } y \in K\}.$$

If X is a topological space we write $X = A|B$ to mean $X = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and A and B are both open in X. In other words $X = A|B$ means $X = A \cup B$ where A and B are nonempty sets which are separated in X (recall that A and B are *separated* provided $\overline{A} \cap B = A \cap \overline{B} = \emptyset$). Sometimes the union of two disjoint sets A, B will be denoted by $A \sqcup B$. We will need the following

two lemmas whose proofs can be found for instance in [Na92, Proposition 6.3] (cf. [Ku68, §46, II, Theorem 4]) and [Na92, Lemma 6.4], respectively.

Lemma 38. *Let X be a connected topological space, C be a connected subset of X and $X \setminus C = A|B$. Then $A \cup C$, $B \cup C$ are connected and, if C is closed, then they are also closed.*

Lemma 39. *Let X be a connected topological space and $a, b \in X$ be such that*

$$X \setminus \{a\} = A_1|A_2 \qquad \textit{and} \qquad X \setminus \{b\} = B_1|B_2 .$$

If $a \in B_1$ and $b \in A_2$ then $B_2 \cup \{b\} \subseteq A_2$.

From Lemma 38 we immediately obtain the following observation which will be used repeatedly.

Lemma 40. *Let X be a connected topological space, C be a connected subset of X and $X \setminus C = A|B$. Then*

(a) *$\partial C \cap \partial A \neq \emptyset$ and $\partial C \cap \partial B \neq \emptyset$;*
(b) *if C is open then A, B are closed and $\partial C \cap A \neq \emptyset$, $\partial C \cap B \neq \emptyset$;*
(c) *if C is closed then A, B are open and $C \cap \partial A \neq \emptyset$, $C \cap \partial B \neq \emptyset$.*

To shorten some formulations in the next lemma and in Proposition 44, we introduce the following terminology. If J is a free interval of a space X and $K \subseteq J$ is an interval such that $J \setminus K$ has two components, we say that K is a *bi-proper subinterval of J*. Note that every point $c \in J$ is a (degenerate) bi-proper subinterval of J.

Lemma 41. *Let X be a connected Hausdorff topological space, J be a free interval of X and K be a subinterval of J such that $X \setminus K$ is not connected. Then for every separation $X \setminus K = L_K|R_K$ the notation can be (and in what follows always will be) chosen in such a way that*

$$L_K \cap J = J_K^- \qquad \textit{and} \qquad R_K \cap J = J_K^+ . \tag{9.1}$$

Moreover, if K is a bi-proper subinterval of J, then J_K^-, J_K^+ are nonempty and L_K, R_K are the (two) components of $X \setminus K$. Regardless of whether K is bi-proper or not, it holds

$$\overline{L_K} = \begin{cases} L_K \cup \{\inf K\} & \textit{if } J_K^- \neq \emptyset \\ L_K & \textit{otherwise} \end{cases} \qquad \textit{and} \qquad \overline{R_K} = \begin{cases} R_K \cup \{\sup K\} & \textit{if } J_K^+ \neq \emptyset \\ R_K & \textit{otherwise.} \end{cases} \tag{9.2}$$

Before reading the proof we recommend to see Example 45. It illustrates that if the subinterval K is not bi-proper then L_K, R_K are not uniquely determined by K and they need not be connected.

Proof. First we prove (9.2) assuming that the rest of the lemma is true. Since L_K and R_K are separated we have $\overline{L_K} \setminus L_K \subseteq K \subseteq J$. If J_K^- is nonempty then, by (9.1) and taking into account that J is open in X, we get $\overline{L_K} \cap J = J_K^- \cup \{\inf K\}$ and so $\overline{L_K} = L_K \cup \{\inf K\}$. If $J_K^- = \emptyset$, $\overline{L_K} \cap J = \emptyset$ and $\overline{L_K} = L_K$. Similarly for $\overline{R_K}$.

It remains to prove the lemma without (9.2).

Since K disconnects X, there are L_K and R_K with $X \setminus K = L_K|R_K$. The sets J_K^-, J_K^+ are connected (possibly empty); if some of these two sets is nonempty, then it is either a subset of L_K or a subset of R_K. Distinguish four cases.

If $K = J$ then there is nothing to prove since *every* separation $X \setminus K = L_K|R_K$ satisfies (9.1). If $J_K^- = \emptyset \neq J_K^+$ then, by changing the notation of the sets L_K and R_K if necessary, we obtain (9.1) and the same argument works if $J_K^+ = \emptyset \neq J_K^-$; in these two cases the first part of the lemma just fixes the notation so that equation (9.1) hold. It remains to prove the lemma if K is bi-proper.

So assume that K is bi-proper (i.e. J_K^-, J_K^+ are nonempty) and $X \setminus K = L_K|R_K$. By changing the notation, if necessary, we may assume that $L_K \supseteq J_K^-$; we are going to prove that $R_K \supseteq J_K^+$. Suppose, on the contrary, that $L_K \supseteq J_K^+$. Then $\partial K \cap \partial R_K \subseteq J \cap \overline{R_K} \subseteq \overline{J \cap R_K} = \emptyset$ (since J is open and $J \subseteq K \cup L_K$) which contradicts Lemma 40(a).

Once we know that $L_K \supseteq J_K^-$, $L_K \cap K = \emptyset$ and $L_K \cap J_K^+ = \emptyset$ (since $R_K \supseteq J_K^+$), then taking into account that $J = J_K^- \sqcup K \sqcup J_K^+$ we get that $L_K \cap J = J_K^-$. Analogously $R_K \cap J = J_K^+$.

To show that L_K is connected, suppose on the contrary that $L_K = A|B$. Since J_K^- is connected, it is a subset of A or B, say $J_K^- \subseteq A$. We claim that $X = B|(A \cup K \cup R_K)$.

Now $B \cap J \subseteq B \cap (A \cup K \cup R_K) = \emptyset$. Since J is open, $\overline{B} \cap J \subseteq \overline{B \cap J}$. Hence, $\overline{B} \cap J = \emptyset$. Since J is homeomorphic to an open interval and $K \subseteq J$ is bi-proper, for $a = \inf K$ and $b = \sup K$ we have $K \subseteq [a,b] \subseteq J$. Obviously, $[a,b]$ is compact and since X is assumed to be Hausdorff, $[a,b]$ is a closed set in X. Thus $\overline{K} \subseteq [a,b] \subseteq J$ and so $\overline{B} \cap \overline{K} \subseteq \overline{B} \cap J = \emptyset$. Using this fact we get $\overline{B} \cap (A \cup K \cup R_K) = \emptyset$ and $B \cap \overline{A \cup K \cup R_K} = \emptyset$. This contradicts the connectedness of X and so we have proved that L_K is connected. Similarly, R_K is connected. □

Lemma 42. *Let X be a connected Hasudorff topological space with a free interval J and let $c \in J$ be a cut point of X. Then*

(a) *$X \setminus \{c\} = L_c|R_c$ where L_c, R_c are the connected sets from Lemma 41;*
(b) *$\overline{L_c} = L_c \cup \{c\}$ and $\overline{R_c} = R_c \cup \{c\}$;*
(c) *$\{c\}$ is closed in X and L_c and R_c are open in X.*
(d) *each $z \in J$ is a cut point of X;*
(e) *every bi-proper subinterval K of J cuts X into exactly two components.*

Proof. (a) follows from Lemma 41.

(b) Using (9.2) from Lemma 41 we get $\overline{L_c} = L_c \cup \{\inf\{c\}\} = L_c \cup \{c\}$ and, similarly, $\overline{R_c} = R_c \cup \{c\}$.

(c) It follows from (b) that $R_c = X \setminus (L_c \cup \{c\}) = X \setminus \overline{L_c}$ is open. Similarly L_c is open and so $\{c\} = X \setminus (L_c \cup R_c)$ is closed.

(d) Fix a point $z \in J \setminus \{c\}$, say $z \prec c$. Then $[z,c) \subseteq J_c^- \subseteq L_c$. Observe also that, since X is Hausdorff, the compact set $[z,c]$ is closed and so we have $\overline{(z,c]} = \overline{[z,c)} = [z,c]$.

Put $L = L_c \setminus [z,c)$ and $R = (z,c] \sqcup R_c$. Then $X = L \sqcup \{z\} \sqcup R$ and we only need to show that L and R are separated. First, $\overline{R} = [z,c] \cup \overline{R_c} = [z,c) \sqcup \overline{R_c}$. Further, we show that $\overline{L} = L \cup \{z\}$. This follows from three facts. First, by (c), $\overline{L} \subseteq \overline{L_c} = L_c \cup \{c\}$. Second, no point $x \in (z,c]$ belongs to $\overline{L}$ since J_z^+ is a neighborhood of x disjoint with L. Third, $z \in \overline{L}$ since every neighborhood of z intersects $J_z^- \subseteq L$. Then for the set $L_c = L \sqcup [z,c)$ we get $\overline{L_c} = \overline{L} \cup [z,c] = \overline{L} \sqcup (z,c]$. So $\overline{L} = \overline{L_c} \setminus (z,c]$. To summarize, $\overline{L} = \overline{L_c} \setminus [z,c)$ and $\overline{R} = [z,c) \sqcup \overline{R_c}$. Hence $\overline{L} \cap R = \left(\overline{L_c} \setminus (z,c]\right) \cap ((z,c] \sqcup R_c) \subseteq \overline{L_c} \cap R_c = \emptyset$ and similarly $L \cap \overline{R} \subseteq L_c \cap \overline{R_c} = \emptyset$, i.e. L, R are separated.

(e) Let K be a bi-proper subinterval of J. To show that $X \setminus K$ has exactly two components it is sufficient, by Lemma 41, to show that $X \setminus K$ is not connected. If K is degenerate this follows from (d). So assume that K is non-degenerate and choose a point a from the interior of K. By (d), a is a cut point of X and, by Lemma 41, the sets L_a, R_a form a separation of $X \setminus \{a\}$. It follows that $(X \setminus K) \cap L_a$ and $(X \setminus K) \cap R_a$ form a separation of $X \setminus K$ and so $X \setminus K$ is not connected. □

Lemma 43. *Let X be a connected compact Hausdorff space and J be a free interval of X. Assume that $X \setminus J$ is disconnected. Then $X \setminus J$ has exactly two components L_J, R_J, the set ∂J is nowhere dense, has exactly two components $\mathcal{L}_0, \mathcal{R}_0$ and the notation can be chosen in such a way that $\mathcal{L}_0 \subseteq L_J$, $\mathcal{R}_0 \subseteq R_J$ and for each $c \in J$ it holds*

$$\overline{J_c^-} = \mathcal{L}_0 \sqcup J_c^- \sqcup \{c\}, \quad \overline{J_c^+} = \{c\} \sqcup J_c^+ \sqcup \mathcal{R}_0 \quad \text{and} \quad X \setminus \{c\} = (L_J \sqcup J_c^-) \mid (J_c^+ \sqcup R_J). \tag{9.3}$$

Proof. Fix a separation $X \setminus J = L_J|R_J$ whose existence is guaranteed by Lemma 41.

Now we show the existence of $\mathcal{L}_0, \mathcal{R}_0$ with the required properties. Since J is a free interval in X we may identify it with the *real interval* $(0,1)$. For $0 < \delta < 1$ put $\mathcal{L}_\delta = (0,\delta)$ and $\mathcal{R}_\delta = (1-\delta, 1)$. Then $\mathcal{L}_0 = \bigcap_{\delta>0} \overline{\mathcal{L}_\delta}$, being the intersection of a nested family of nonempty compact connected sets in a compact Hausdorff space, is a nonempty compact connected set. By changing the notation of $\mathcal{L}_\delta, \mathcal{R}_\delta$, if necessary, we may assume that $\mathcal{L}_0 \subseteq L_J$. Further, $\mathcal{L}_0 \subseteq \overline{J} \setminus J$; indeed, the inclusion $\mathcal{L}_0 \subseteq \overline{J}$ is trivial and a point from J cannot belong to $\mathcal{L}_0$ since J is open. Similarly for $\mathcal{R}_0 = \bigcap_{\delta>0} \overline{\mathcal{R}_\delta}$. Hence $\mathcal{L}_0 \cup \mathcal{R}_0 \subseteq \overline{J} \setminus J$. Also the converse inclusion holds since for every $\delta > 0$ we have $\overline{J} = \overline{\mathcal{L}_\delta} \cup [\delta, 1-\delta] \cup \overline{\mathcal{R}_\delta}$ and so $\overline{J} \setminus J \subseteq \bigcap_\delta (\overline{\mathcal{L}_\delta} \cup \overline{\mathcal{R}_\delta}) = \mathcal{L}_0 \cup \mathcal{R}_0$. We have thus proved that

$$\partial J = \overline{J} \setminus J = \mathcal{L}_0 \cup \mathcal{R}_0.$$

By Lemma 40(b), each of the sets L_J, R_J intersects $\partial J = \mathcal{L}_0 \cup \mathcal{R}_0$. Since $\mathcal{L}_0 \subseteq L_J$ we must have $\mathcal{R}_0 \subseteq R_J$. It follows that $\mathcal{L}_0, \mathcal{R}_0$ are disjoint which implies that they are the two components of ∂J. The boundary ∂J is nowhere dense since J is open.

Now we prove that L_J, R_J are connected. Suppose that L_J is not connected and fix a separation $A|B$ of L_J. Then the connected set $\mathcal{L}_0$ lies in one of the sets A, B, say $\mathcal{L}_0 \subseteq B$. Then A, J are separated and so $X = A|(B \cup J \cup R_J)$ is not connected, which is a contradiction. Analogously we can show that R_J is connected.

Fix $c \in J$. It remains to show (9.3). For every sufficiently small δ we have $J_c^- \supseteq \mathcal{L}_\delta$, so $\overline{J_c^-} \supseteq \bigcap_{\delta>0} \overline{\mathcal{L}_\delta} = \mathcal{L}_0$. Hence $\overline{J_c^-} \supseteq \mathcal{L}_0 \sqcup J_c^- \sqcup \{c\}$. Further, for every sufficiently small $\delta > 0$ the set $\overline{\mathcal{L}_\delta} \cup J_c^- \cup \{c\} = \overline{\mathcal{L}_\delta} \cup [\delta, c]$ is closed and contains J_c^-, hence it contains $\overline{J_c^-}$. Thus $\overline{J_c^-} \subseteq \bigcap_{\delta>0} \left(\overline{\mathcal{L}_\delta} \cup J_c^- \cup \{c\}\right) = \mathcal{L}_0 \sqcup J_c^- \sqcup \{c\}$. We have proved the first formula in (9.3); analogously for the second one. The third formula follows from the first two ones and the fact that L_J, R_J are separated and closed. □

Proposition 44. *Let X be a connected Hausdorff topological space and J be a free interval in X. Consider the following conditions:*

- (1a) *$X \setminus J$ is not connected;*
- (1b) *$X \setminus J$ has exactly two components;*
- (2a) *there is a point $x \in J$ which cuts X;*
- (2b) *every point $x \in J$ cuts X into exactly two components (i.e. J is a disconnecting interval, see Definition 2);*
- (3a) *there is a bi-proper subinterval K of J such that the set $X \setminus K$ is not connected;*
- (3b) *for every bi-proper subinterval K of J the set $X \setminus K$ has exactly two components;*
- (4a) *there is a subinterval K of J such that the set $X \setminus K$ is not connected;*
- (4b) *for every subinterval K of J the set $X \setminus K$ has exactly two components.*

Then the following hold:

- (i) *(4b) ⇒ (1b) ⇒ (1a) ⇒ (4a) and (4b) ⇒ (2a) ⇔ (2b) ⇔ (3a) ⇔ (3b) ⇒ (4a).*
- (ii) *If X is a* compact *connected Hausdorff space then all the eight conditions (1a) – (4b) are equivalent.*

In the case (i) no other implication, except of those which follow by transitivity, is true.

Proof. (i) It is sufficient to show (2a)⇔(2b)⇔(3a)⇔(3b) since all other needed implications are trivial. Since the implications (3b)⇒(2b)⇒(2a)⇒(3a) are also trivial, it remains to prove that (3a)⇒(3b). However, (2a)⇒(3b) holds by Lemma 42 and so it is sufficient to prove (3a)⇒(2a). To this end, let $K \subseteq J$ be a bi-proper subinterval of J such that $X \setminus K$ is not connected. If K is degenerate there is nothing to prove. Otherwise put $a = \inf K$ and $b = \sup K$. Then, since X is Hausdorff, the same argument as in the last paragraph of the proof of Lemma 41 gives that $[a, b]$ is a closed set in X and so $\overline{K} = [a, b]$. By Lemma 41, $X \setminus K = L_K|R_K$ where L_K, R_K are connected and contain J_K^-, J_K^+, respectively. We are going to prove that a is a cut point of X. We do not know whether a belongs to L_K or to K, therefore denote $L_K' = L_K \setminus \{a\}$ and $K' = K \setminus \{a\}$. Obviously, $\overline{K'} = \overline{K} = [a, b]$ and $\overline{L_K'} = \overline{L_K} = L_K \cup \{a\}$ by Lemma 41. It follows that $\overline{K'} \cap \overline{L_K'} = \overline{K} \cap \overline{L_K} = \{a\}$ and hence $\overline{K'} \cap L_K' = K' \cap \overline{L_K'} = \emptyset$. Using these facts and the fact that L_K and R_K are separated we immediately get that $X \setminus \{a\} = L_K'|(K' \sqcup R_K)$.

(ii) Assume that X is also compact. We need to prove that (4a)⇒(4b). The implications (1a)⇒(1b) and (1a)⇒(2a) follow from Lemma 43. Moreover, we know that (2a) is equivalent with (3b). So to finish the proof it is sufficient to show that (4a)⇒(1a) and that (1b) and (3b) together imply (4b).

First we show that (4a)⇒(1a). So assume (4a). We may identify J with the real interval $(0, 1)$. Write $X \setminus K = L_K|R_K$ as in Lemma 41. We are going to show that $K' = K \cup J_K^-$ also cuts X. This is trivial if $J_K^- = \emptyset$, so assume that $J_K^- \neq \emptyset$. Put $a = \inf K > 0$. Take a decreasing sequence $a > a_1 > a_2 > \dots$ of real numbers converging to 0. Put $K_n = (a_n, a] \cup K$ and $\mathcal{L}_n = L_K \setminus (a_n, a]$.

Then $\mathcal{L}_n$ is nonempty and trivially for the interval $K_n \subseteq J$ we have $X \setminus K_n = \mathcal{L}_n | R_K$. Since $\mathcal{L}_n \cap J = J^-_{K_n}$, the sets $\mathcal{L}_n$ and R_K in this separation correspond to the sets L_{K_n} and R_{K_n} from Lemma 41. Hence $\overline{\mathcal{L}_n} = \mathcal{L}_n \cup \{\inf K_n\} = \mathcal{L}_n$. Since $(\mathcal{L}_n)_{n=1}^\infty$ is a nested sequence of nonempty compact sets, the set $\mathcal{L} = \bigcap_n \mathcal{L}_n$ is nonempty. Now, using the fact that $K' = \bigcup_n K_n$, we have that $X \setminus K' = \mathcal{L} | R_K$, i.e. K' cuts X. Once we know that $K' = K \cup J^-_K$ cuts X, by an analogous argument we get that also $J = K' \cup J^+_{K'}$ cuts X, i.e. we get (1a).

To show that (1b) and (3b) together imply (4b), fix a subinterval K of J. We need to show that $X \setminus K$ consists of two components. This is trivial if $K = J$ (use (1b)) or if K is bi-proper (use (3b)). It remains to consider the case when $J^-_K = \emptyset$ and $J^+_K \neq \emptyset$ or conversely. Suppose we are in the former case. By (1b), $X \setminus J = L_J | R_J$, where L_J and R_J are connected. We claim that L_J and $J^+_K \sqcup R_J$ are the two components of $X \setminus K$. Indeed, the sets L_J and $J^+_K \sqcup R_J$ are separated (use that for $c \in K$, $\overline{J^+_K} \subseteq \overline{J^+_c}$ and Lemma 43 gives $\overline{J^+_c} \cap L_J = \emptyset$). Further, we know that L_J is connected. Finally, the sets J^+_K and R_J are connected and not separated (if $d \in J^+_K$ then $\overline{J^+_K} \supseteq \overline{J^+_d}$ and, by Lemma 43, $\overline{J^+_d} \cap R_J \supseteq \mathcal{R}_0 \neq \emptyset$). Hence also the set $J^+_K \sqcup R_J$ is connected.

To finish the proof of the proposition we need to show that in the case (i) no other implication, except of those which follow by transitivity, is true. It is easy to see that this requires three counterexamples. We collect them below (see Examples 45–47). $\square$

Example 45 (*In Proposition 44(i), (1a)$\not\Rightarrow$(1b)*). Consider the topologist's sine curve with the interval of convergence replaced by just two points of it. If J is the maximal free interval of this space then $X \setminus J$ consists of three degenerate components.

Example 46 (*In Proposition 44(i), (1b)$\not\Rightarrow$(2a)*). Consider the (non-compact) subspace X of the Euclidean plane defined by

$$X = J \sqcup C_{-1} \sqcup C_1$$

where $C_i = [-1, 1] \times \{i\}$ for $i \in \{-1, 1\}$ and J is the graph of the function $x \mapsto x \cdot \sin(1/1 - |x|)$, $x \in (-1, 1)$. Then X is connected, J is a free interval and $X \setminus J = C_{-1} \cup C_1$ has two components. However, no point $x \in X$ is a cut point of X.

Example 47 (*In Proposition 44(i), (3b)$\not\Rightarrow$(1a)*). Let $X = (0, 1)$ or $X = [0, 1)$ be a subset of the real line. Then $J = (0, 1)$ is a disconnecting interval of X and $X \setminus J$ is connected.

In Proposition 44 we have assumed that X is Hausdorff. Without this assumption we would not have the equivalence of the four conditions (2a), (2b), (3a), (3b). For instance the space in the following example satisfies (2a), (3a) but does not satisfy (2b), (3b).

Example 48. Let X be the line with two origins, i.e. the factor space of $\mathbb{R} \times \{-1, 1\}$ obtained by identifying $x \times (-1)$ with $x \times 1$ for every $x \neq 0$. Let $p : \mathbb{R} \times \{-1, 1\} \to X$ be the corresponding factor map. Then $J = p((-1, 1) \times 1)$ is a free interval containing a cut point of X, e.g. $c = p(1/2 \times 1)$. But $z = p(0 \times 1) \in J$ is not a cut point of X. So we have (2a) but not (2b). Also, $K = p((-1/2, 1/2) \times 1)$ is a bi-proper subinterval of J such that $X \setminus K$ is not connected but has precisely three components. So we have (3a) but not (3b).

References

[ALM00] LL. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, Second edition, Advanced Series in Nonlinear Dynamics, 5, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[AKLS99] Ll. Alsedà, S. Kolyada, J. Llibre, Ľ. Snoha, *Entropy and periodic points for transitive maps*, Trans. Amer. Math. Soc. **351** (1999), no. 4, 1551–1573.

[Ba01] S. Baldwin, *Entropy estimates for transitive maps on trees*, Topology **40** (2001), no. 3, 551–569.

[Ba97] J. Banks, *Regular periodic decompositions for topologically transitive maps*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 505–529.

[BC92] L. S. Block, W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, 1513. Springer-Verlag, Berlin, 1992.

[BDHSS09] F. Balibrea, T. Downarowicz, R. Hric, Ľ. Snoha, V. Špitalský, *Almost totally disconnected minimal systems*, Ergodic Theory Dynam. Systems **29** (2009), no. 3, 737–766.

[BHS03] F. Balibrea, R. Hric, Ľ. Snoha, *Minimal sets on graphs and dendrites*, Dynamical systems and functional equations (Murcia, 2000), Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), no. 7, 1721–1725.

[Bl84] A. M. Blokh, *On transitive mappings of one-dimensional branched manifolds. (Russian)*, Differential-difference equations and problems of mathematical physics (Russian), 3–9, 131, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1984.

[Bl86] A. M. Blokh, *Dynamical systems on one-dimensional branched manifolds. I (Russian)*, Teor. Funktsii Funktsional. Anal. i Prilozhen. **46** (1986), 8–18; translation in J. Soviet Math. **48** (1990), no. 5, 500–508.

[CE80] P. Collet, J. P. Eckmann, *Iterated maps on the interval as dynamical systems*, Progress in Physics, 1. Birkhäuser, Boston, Mass., 1980.

[dMvS93] W. de Melo, S. van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 25, Springer-Verlag, Berlin, 1993.

[D32] A. Denjoy, *Sur les courbes definies par les équations différentielles à la surface du tore* (French), J. Math. Pures Appl. **11** (1932), 333–375.

[DY02] T. Downarowicz, X. Ye, *When every point is either transitive or periodic*, Colloq. Math. **93** (2002), no. 1, 137–150.

[HKO11] G. Harańczyk, D. Kwietniak and P. Oprocha, *A note on transitivity, sensitivity and chaos for graph maps*, J. Difference Eq. Appl. **17** (2011), no. 10, 1549–1553.

[Il00] A. Illanes, *A characterization of dendrites with the periodic-recurrent property*, Topology Proc. **23** (1998), Summer, 221–235.

[Ka88] K. Kawamura, *A direct proof that each Peano continuum with a free arc admits no expansive homeomorphisms*, Tsukuba J. Math. **12** (1988), no. 2, 521–524.

[Ki58] S. Kinoshita, *On orbits of homeomorphisms*, Colloq. Math. **6** (1958), 49-53.

[KS97] S. Kolyada, Ľ. Snoha, *Some aspects of topological transitivity – a survey*, Iteration theory (ECIT 94) (Opava), 3–35, Grazer Math. Ber., **334**, Karl Franzens-Univ. Graz, Graz, 1997.

[KST01] S. Kolyada, Ľ. Snoha, S. Trofimchuk, *Noninvertible minimal maps*, Fundamenta Mathematicae **168** (2001), 141–163.

[Ku68] K. Kuratowski, *Topology. Vol. II*, Academic Press, New York-London, Państwowe Wydawnictwo Naukowe Polish Scientific Publishers, Warsaw, 1968.

[Kw11] D. Kwietniak, *Weak mixing implies mixing*, preprint, 2011.

[Ma11] P. Maličký, *Backward orbits of transitive maps* J. Difference Equ. Appl., to appear.

[MS07] J. Mai, E. Shi, *The nonexistence of expansive commutative group actions on Peano continua having free dendrites*, Topology Appl. **155** (2007), no. 1, 33–38.

[MS09] J.-H. Mai, S. Shao, $\overline{R} = R \cup \overline{P}$ *for graph maps* J. Math. Anal. Appl. **350** (2009), no. 1, 9–11.

[Na92] S. B. Nadler, *Continuum theory. An introduction*, Monographs and Textbooks in Pure and Applied Mathematics, 158, Marcel Dekker, Inc., New York, 1992.

[Si92] S. Silverman, *On maps with dense orbits and the definition of chaos*, Rocky Mountain J. Math. **22** (1992), no. 1, 353–375.

[SW09] E. Shi, S. Wang, *The ping-pong game, geometric entropy and expansiveness for group actions on Peano continua having free dendrites*, Fund. Math. **203** (2009), no. 1, 21–37.

[XYZH96] J. C. Xiong, X. D. Ye, Z. Q. Zhang, J. Huang, *Some dynamical properties of continuous maps on the Warsaw circle* (Chinese), Acta Math. Sinica (Chin. Ser.) **39** (1996), no. 3, 294–299.

[Ye92] X. D. Ye, *D-function of a minimal set and an extension of Sharkovskii's theorem to minimal sets*, Ergodic Theory Dynam. Systems **12** (1992), no. 2, 365–376.

[ZPQY08] G. Zhang, L. Pang, B. Qin, K. Yan, *The mixing properties on maps of the Warsaw circle*, J. Concr. Appl. Math. **6** (2008), no. 2, 139–144.

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